A One-Step Integral Collocation-Homotopy Analysis Method for Solutions of Non-linear Integro-Differential Equations

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Abstract
This paper presents an effective algorithm which is a combination of integral collocation method and homotopy analysis method that can be used for effective analytical and numerical solutions of nonlinear initial and boundary value problems of integro-differential equations. Examples were given to verify the efficiency and reliability of the new algorithm and comparisons were made between the obtained results and those in the literature. Our numerical results confirmed that the new algorithm provides relatively accurate results with rapid rate of convergence than the standard homotopy analysis method.
Keywords: Integral Collocation, Homotopy Analysis Method, Nonlinear Integro-differential Equations, Algorithm, Initial Value Problems, Boundary Value Problems.

Introduction
Recently, considerable interest in analytical and numerical solutions of integro-differential equations has been stimulated due to their various applications in many fields like Physics, astronomy, potential theory, fluid dynamics, biological models and chemical kinetics, control theory, elasticity, polymer rheology, etc. It is usually difficult to obtain the closed-form solutions of two-point boundary value problems of integro-differential equations, therefore the need to solve the problems by approximate and numerical methods.

In recent times, several numerical methods of solving integro-differential equations have been developed such as Collocation method in Ordokhani and Dehestani (2013), Sweilam et al. (2012) and Yousefi and Razzaghi (2005). Homotopy analysis method in Ahmad et al. (2012). Reproducing Kernel Hilbert Space (RKHS)method in Mohammed et al. (2012). Pseudo spectral method in Sweilam et al. (2013). Variational iteration method in Khader (2012), Sweilam et al. (2013) and Sweilam (2007). Homotopy perturbation method in Khader (2012b) and Sweilam et al. (2008). Spline function, expansion in Mohammed and Khader (2011). New Iterative method in Hemeda (2012). Bernstein polynomial method in Yadollah and Sara (2011). Legendre Wavelet method in Venkatesh et al. (2013) and Yousefi and Razzaghi (2005). Spectral-homotopy analysis method in Atabakan et al. (2012) and so on.

Homotopy Analysis Method (HAM) was first proposed by Liao (1992). It is based on the homotopy concept in topology for solving nonlinear differential equations. Unlike the traditional perturbation methods like Lyapunov's, artificial small parameter method in Lyapunov (1992). Adomian decomposition methods in Adomian (1994) and Adomian (1990) and the δ-expansion method, which are the special cases of HAM, this approach does not need a small perturbation parameter.

In this work, integral collocation method is blended with homotopy analysis method and this is done by using integrated Chebyshev polynomials to represent the initial approximation and the derivatives corresponding to m = 1. Thereafter, the first order approximation is obtained which is then substituted into the original equation to obtain residual equation. The residual equation is then collocated at the chosen collocation points and also the given conditions are used to determine the unknown constants which are later substituted into the first order approximation to obtain the required solution. Thus, we consider integro-differential equation of the form:

$$\sum_{k=o}^{n} P_k(x)y^{(k)}(x) = g(x) + \lambda_1 \int_a^b k_1(x,t)\psi_1(t,y(t))\lambda_2 \int_a^x k_2(x,t)\psi_2(t,y(t))dt, \quad 0 \le a \le x,t \le b \quad (1)$$

under the mixed conditions

$$\sum_{k=o}^{n-1}\left[a_{jk}\, y^{(k)}(a) + b_{jk} y^{(k)}(b) \right] = \lambda_j,\ j = 0,1,\cdots,n-1 \quad (2)$$

where y(x) is an unknown function, the known function, are $P_k(x)$, k = 0, 1, …, n, g(x), $k_1(x,t)$, $k_2(x,t)$, ψ_1 (t, y(t)) and ψ_2 (t, y (t)). Also, a_{jk}, b_{jk}, λ_1, λ_2 and λ_j are real or complex constants.

The Homotopy Analysis Method

Homotopy analysis method is a non-perturbation method which was proposed by Liao (1992) in his Ph. D thesis. The method is valid for strongly nonlinear problems with or without small / large parameters. The homotopy analysis method constructs a sequence which continuously deforms from an initial guess of the solution of a differential equation to the exact solution.

Let us consider the following differential equation

$$N[\,y(x)] = 0 \quad (3)$$

where N is a nonlinear operator for this problem, x denotes an independent variables, y (x) is an unknown function. By means of generalizing the traditional homotopy method, Liao (1992) constructs the so called zero-order deformation equation

$$(1 - q)\, L\, [\, \phi\,(x; q) - y_0(x)\,] = q\mathbb{C}_0\, H(x)\, N\, [\phi\,(x;q] \quad (4)$$

where q ∈ [0, 1] is the embedding parameter, $\mathbb{C}_0 \neq 0$ is a non -zero auxiliary parameter, H(x) ≠ 0 is an auxiliary function, L is an auxiliary linear operator, $y_o(x)$ is an initial guess of y(x), y(x; q), is an unknown function, on the independent variables x and p.

Obviously, when q = 0 and q = 1, it holds

$$\phi(x;0) = y_0(x), \quad \phi(x;1) = y(x) \quad (5)$$

respectively. Thus as q increases from 0 to 1, the solution $y(x;q)$ varies from the initial guess $y_0(x)$ to the solution y(x). Expanding y (x,p) in Taylor series with respect to q, we have

$$\phi(x;p) = y_0(x) + \sum_{m=1}^{+\infty} y_m(x)q^m, \quad (6)$$

where

$$y_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x:q)}{\partial q^m}\bigg|_{q=0} \tag{7}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter, c_0 and the auxiliary function H(x) are selected such that the series (6) is convergent at q =1, then due to (5), we have

$$y(x) = y_o(x) + \sum_{m=1}^{\infty} y_m(x) \tag{8}$$

Define the vector

$$\vec{y}_n = \{y_o(x), y_1(x), \cdots, y_n(x)\} \tag{9}$$

Differentiating (4) m-times with respect to the embedding parameter q and then setting q = 0 and finally dividing by m!, we have the so–called mth–order deformation equation

$$L[y_m(x) = \chi_m y_{m-1}(x)] = c_o H(x) R_m(\vec{y}_{m-1}(x)), \tag{10}$$

where

$$R_m(\vec{y}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x;q)]}{\partial p^{m-1}}\bigg|_{q=0} \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{12}$$

For any given nonlinear operator N, the term $R_m(\vec{y}_{m-1})$ can be easily expressed by (11). Thus, we can gain $y_0(x)$, $y_1(x)$, $y_2(x)$,. . ., by means of solving (10) one after the order. The mth–order approximation of y(x) is given by $y(x) = \sum_{k=0}^{m-1} y_k(x)$

Construction Process of Integral Collocation-Homotopy Analysis Method (IC-HAM)

This section presents an algorithm based on integral collocation and homotopy analysis methods to solve (1) and (2). The zeroth–order deformation equation is given by

$$(1-q)L[Y(x;q) - y_0(x)] = qc_o H(\varepsilon)[N(Y(\varepsilon;q))], \quad q \in [0,1] \tag{13}$$

where

$$N[Y(\varepsilon;q)] = \sum_{k=0}^{n} P_k(\varepsilon) y^{(k)}(\varepsilon) - g(\varepsilon) - \lambda_1 \int_a^b k_1(\varepsilon,t)\psi_1(t,y(t))dt$$
$$-\lambda_2 \int_o^x k_2(\varepsilon,t)\psi_2(t,y(t))dt, \tag{14}$$

$$H(\varepsilon) = \frac{(x-\varepsilon)^{k-1}}{(k-1)!} \qquad and \qquad L^{-1} = \int_o^x (\bullet)d\varepsilon$$

The initial guess $y_0(x)$ of the solution y(x) and other derivatives corresponding to m = 1 are determined as follows:

$$\frac{d^k y_o(x)}{dx^k} = \sum_{n=o}^{N} a_n T_n(x) \tag{15}$$

Integration equation (15) successively, gives

$$\begin{aligned}\frac{d^{k-1} y_o(x)}{dx^{k-1}} &= \sum_{n=o}^{N} a_n \int T_n(x) \quad dx + c_1 \\ &= \sum_{n=o}^{N+1} \delta_{n,k-1} \phi_n^{[k-1]}(x)\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{d^{k-2} y_o(x)}{dx^{k-2}} &= \sum_{n=o}^{N} a_n \int \phi_n^{[k-1]}(x) \quad dx + c_1 x + c_2 \\ &= \sum_{n=o}^{N+2} \delta_{n,k-2} \phi_n^{[k-2]}(x)\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{dy_o(x)}{dx} &= \sum_{n=o}^{N} a_n \int \varphi_n^{[2]}(x) \quad dx + c_1 \frac{x^{k-2}}{(k-3)!} + c_2 \frac{x^{k-3}}{(k-\partial)!} + \cdots + c_{k-2} x + c_k \\ &= \sum_{n-0}^{N+k-1} \delta_{n,1} \varphi_n^{[1]}(x) \\ &\vdots\end{aligned} \tag{18}$$

$$\begin{aligned}y_0(x) &= \sum_{n=o}^{N} a_n \int \phi_n^{[2]}(x) \quad dx + c_1 \frac{x^{k-1}}{(k-1)!} + c_2 \frac{x^{k-2}}{(k-2)!} + \cdots + c_{k-1} x + c_k \\ &= \sum_{n=o}^{N+k} \delta_{n,0} \phi_n^{[0]}(x)\end{aligned} \tag{19}$$

where $T_n(x)$ are Chebyshev polynomials of degree n of the first kind which is valid in the interval $-1 \leq x \leq 1$ and is given by

$$T_n(x) = \cos(\mathrm{n} \cos^{-1} x) \tag{20}$$

and which satisfy the recurrence relation given by

$$T_{n+1}(x) = 2xT_n(x) = T_{n-1}(x), \quad n \geq 1 \tag{21}$$

Following the HAM procedure, we formulate the high-order deformation equations by differentiating the zeroth-order deformation equation m times with respect to q then dividing by m! to get

$$L\left[y_m(x) - \chi_m y_{m-1}(x)\right] = c_o H(\varepsilon) R_m(\vec{y}_{m-1}(\varepsilon)), \tag{22}$$

Operating the operator L^{-1}, the inverse of $\frac{d}{d\varepsilon}$ to both sides of (22), then the mth-order deformation will have the following form:

$$y_m(m) = \chi_m y_{m-1}(x) + c_o L^{-1}(H(\varepsilon) R_m(\vec{y}_{m-1}(\varepsilon)), \tag{23}$$

where

$$R_m(\vec{y}_{m-1}(\varepsilon)) = \sum_{k=0}^{n} P_k(\varepsilon) y^{(k)}(\varepsilon) - \lambda_1 \int_a^b k_1(\varepsilon,t)\psi_1(t,y(t))dt$$
$$-\lambda_2 \int_o^x k_2(\varepsilon,t)\psi_2(t,y(t))dt - (1-\chi_m)g(\varepsilon) \tag{24}$$

Thus, we have the following recursive formula for the IC-HAM:

$$y_0(x) = \sum_{n=0}^{N+k} \delta_{n,0}\phi_n^{[0]}(x)$$
$$y_1(x) = c_0 L^{-1}(H(\varepsilon)R_1(\vec{y}_0(\varepsilon)) \tag{25}$$
$$y_m(x) = y_{m-1}(x) + c_0 L^{-1}(H(\varepsilon)R_m(\vec{y}_{m-1}(\varepsilon)), \qquad m = 2,3,\ldots$$

Moreover, substituting (8) into (1) yields the residual equation:

$$R(x,a,c) = \sum_{k=0}^{n} P_k(x)\frac{d^k}{dx^k}(y_o(x) + \sum_{m=1}^{\infty} y_m(x))$$
$$= g(x) + \lambda_1 \int_a^b k_1(x,t)\psi_1(\mathrm{t},(\mathrm{y}_0(\mathrm{t}) + \sum_{m=1}^{\infty} y_m(t))\,dt + \lambda_2 \int_a^x k_2(x,t)\psi_2(t,(y_0(t) + \sum_{m=1}^{\infty} y_m(t))\,dt \tag{26}$$

The residual equation (26) is thereafter collocated at the point $x = x_j$ which gives

$$\left.\begin{aligned} R(x_j,a,c) &\Rightarrow \sum_{k=0}^{n} P_K(x_j)\frac{d^k}{dx_j^{\,k}}\left(y_0(x_j) + \sum_{m=1}^{\infty} y_m(x_j)\right) \\ &= g(x_j) + \lambda_1 \int_a^b k_1(x_j,t)\psi(t,(y_0(t) + \sum_{m=1}^{\alpha} y_m(t)))\,\mathrm{d}t \\ &+ \lambda_2 \int_a^{x_y} k_2(x_j,t)\psi_2(t,(y_0(t) + \sum_{m=1}^{\alpha} y_m(t)))dt \end{aligned}\right\} \tag{27}$$

where

$$x_j = a + \frac{(b-a)_j}{N+2}, \qquad j = 1,2,3,\ldots,\mathrm{N}+1 \tag{28}$$

Thus, (27) gives (N+1) nonlinear algebraic system of equations in (N+k+1) unknown constants. Extra k equations are then obtained from the boundary conditions. Altogether, we have (N+k+1) algebraic nonlinear system of equations. These (N+k+1) nonlinear equations are then solved by Newton`s method to obtain the (N+k+1) unknown constants which are then substituted into the first-order approximation ($y_0(x) + y_1(x)$).
It should be noted that the convergence- control parameter, c_0, is taking to be 1 for this proposed method.

Numerical Examples

To show the efficiency of Integral Collocation-Homology Analysis Method (IC.-HAM) described above, six examples are considered. The exact solutions of most of the problems were gotten from Babolian et al. (2009a), Babolian et al. (2009b), Maleknejad et al. (2011), Venkatesh et al. (2013) and Ordokhani and Dehestani (2013)

Example 1: Consider the first–order nonlinear Fredholm integro-differential

equation $y'(x) = 1 - \frac{1}{3}x + \int_0^1 xy^2(t)dt, \quad 0 \le x \le 1$

with the initial condition y(0) = 0 and exact solution $y(x) = x$.

According to (4), the zeroth-order deformation can be given by

$$(1-q)\mathrm{L}[y(x;q) - y_0(x)] = qc_0\mathrm{H}(\varepsilon)\left[\frac{dy(\varepsilon;q)}{\cdot} - (1-\frac{1}{3}\varepsilon) - \int_0^1 \varepsilon y^2(t)\,\mathrm{d}t\right] \tag{29}$$

The initial approximation has the form of (19) and we chose the auxiliary linear operator

$$(y(\varepsilon,q) = L\frac{dy(\varepsilon,q)}{dx} \tag{30}$$

and

$$\mathrm{H}(\varepsilon) = 1$$

Hence, the mth-order deformation is given by

$$\mathrm{L}[y_m - \chi_m y_{m-1}(x)] = H(\varepsilon)R_m(\vec{y}_{m-1}(\varepsilon)) \tag{31}$$

where

$$\mathrm{R}_1(\vec{y}_o(\varepsilon)) = \frac{dy_0(\varepsilon;q)}{d\varepsilon} - (1-\frac{1}{3}\varepsilon) - \int_0^1 \varepsilon y_0^2(t)dt$$

and

$$R_m(\vec{y}_{m-1}(\varepsilon)) = \frac{dy_{m-1}(\varepsilon;q)}{d\varepsilon} - \int_0^1 \varepsilon y_0^2(t)dt - (1-\chi_m)(1-\frac{1}{3}\varepsilon); \qquad m \ge 2$$

Now, the solution of the mth-order deformation equation for m ≥ 1 becomes

$$\mathrm{y_m}(x) = \chi_m y_{m-1}(x) + \int_0^x R_m(\vec{y}_{m-1}(\varepsilon))d\varepsilon \tag{32}$$

Consequently, the first few terms of the IC–HAM series solution for N = 1 are as follows:

$$y_0(x) = c_1 + (a_0 - a_1)x + a_1x^2$$

$$\mathrm{y_1}(x) = (\frac{1}{12}a_1a_0 + a_1 + \frac{1}{6} - \frac{1}{60}a_1^2 - \frac{1}{2}c_1^2 + \frac{1}{6}c_1a_1 - \frac{1}{6}a_0^2 - \frac{1}{2}c_1a_0)x^2 + (a_0 - a_1 - 1)x$$

and so on

The first–order approximate solution by IC-HAM is

$$y(x) = y_0(x) + y_1(x)$$

that is,

$$\mathrm{y(x)} = (2a_1 - \frac{1}{60}a_1^2 + \frac{1}{12}a_1a_0 + \frac{1}{6} - \frac{1}{2}c_1a_0 - \frac{1}{2}c_1^2 + \frac{1}{6}c_1a_1 - \frac{1}{6}a_0^2)x^2 + (2a_0 - 2a_1 - 1)x + c_1 \tag{33}$$

and the residual of the solution is

$$\mathrm{R} = \frac{dy(x)}{d\,x} - (1-\frac{1}{3}x) - \int_0^1 xy^2(t)dt \tag{34}$$

Collocating (34) at the points $x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}$ and using the given condition, we obtained system of nonlinear algebraic equations which were solved by using Newton's method. Thus, we get

$$a_0 = 1, \quad a_1 = 0 \quad and \quad c_1 = 0$$

Substituting these values into (33), we obtain $y(x) = x$, which is the exact solution.

Example 2: Consider the nonlinear FVIDE

$$y'(x) + y(x) = 2x + x^2 + \frac{1}{10}x^6 - \frac{1}{32} + \frac{1}{4}\int_0^1 ty^3(t)dt - \frac{1}{2}\int_0^x xy^2(t)dt, \quad 0 \le x \le 1,$$

with condition y(0) = 0. The exact solution to this example is $y(x) = x^2$

Following the same procedure as discussed in Example 1 for case N = 2, we obtained $a_0 = 1, \ a_1 = 1, \ a_2 = 0, \ and \ c_1 = 0$. Substituting these values into the first- order approximation, we obtain

$$y(x) = x^2 - 1X10^{-11}x^5 \tag{35}$$

which is very close to the exact solution $y(x) = x^2$

Example 3: Consider the nonlinear VHIDE

$$y'(x) = -2sin(x) - \frac{1}{3}cos(x) - \frac{2}{3}cos(2x) + \int_0^x cos(x-t)\,\mathrm{y}^2\, dt, \quad 0 \le x \le 1,$$

with condition y(0) = 1. The exact solution to this equation is $y(x) = \cos(x) - \sin(x)$. For each test point, the absolute error between the exact solution and the results obtained by the Bessel collocation method and Hybrid Legendre polynomials and block–pulse functions approach and IC-HAM is compared in Table 1. With only one iteration for N = 4 a better approximation has been obtained than the results in Maleknejad et al. (2011) and Ordokhani and Dehestani (2013).

Table 1: Absolute errors of the first order approximate solution

X	**Exact solution**	**IC-HAM solution N = 4**	**IC-HAM Error N = 4**	**Method of Ordokhani and Dehestani (2013) N = 4**	**Method of Maleknejad et al. (2011) n = 8 , m = 8**
0	1.0000000000	1.0000000000	0.00	0.00	1.00 E – 06
0.1	0.8951707486	0.8951626483	8.10 E – 06	1.07 E – 05	1.60 E – 05
0.2	0.7813972470	0.7813883438	8.90 E – 06	3.26 E – 05	2.56 E – 04
0.3	0.6598162824	0.6598074590	8.82 E – 06	7.65 E – 05	8.40 E – 05
0.4	0.5316426517	0.5316333622	9.29 E – 06	1.98 E – 04	9.43 E – 04
0.5	0.3981570233	0.3981470954	9.93 E – 06	5.09 E – 04	1.20 E – 05
0.6	0.2606931415	0.2606827164	1.04 E – 05	1.19 E – 03	2.76 E – 04
0.7	0.1206245001	0.1206132468	1.13 E – 05	2.50 E – 03	4.70 E – 05
0.8	-0.0206493816	-0.02066277097	1.34 E – 05	4.82 E – 03	1.10 E – 05
0.9	-0.1617169413	-0.1617339766	1.70 E – 05	8.60 E – 05	7.80 E – 05
1.0	-0.3011686789	-0.3011877744	1.91 E – 05	1.44 E – 02	8.15 E – 04

Example 4: Consider the second-order nonlinear integro-differential equation

with the initial condition: $y^{//}(x) = 2 - \frac{x}{2} + \int_0^1 xy(t)y^/(t)dt$

$y(0) = y^/(0) = 0$

Similarly, the first- order approximate solution by IC- HAM for this example when N = 2 is

$$y(x) = \frac{4}{3}a_2x^4 + (-\frac{1}{12}c_2a_0 - \frac{1}{216}a_2a_1 + \frac{1}{36}c_1a_2 + \frac{1}{72}a_2a_0$$
$$+\frac{1}{12} + \frac{1}{36}c_1a_1 + \frac{1}{72}a_0a_1 + \frac{2}{3}a_1 - \frac{1}{6}c_2c_1 - \frac{8}{3}a_2 - \frac{1}{12}c_1{}^2$$
$$+\frac{1}{36}c_2a_2 - \frac{1}{12}c_1a_0 + \frac{1}{36}c_2a_1 - \frac{1}{432}a_2{}^2 - \frac{1}{432}a_1{}^2 - \frac{1}{48}a_0{}^2)x^3$$
$$+(-a_1 + a_0 + a_2 - 1)x^2 + c_1x + c_2 \qquad (36)$$

and the residual of the solution is

$$R = \frac{d^2y(x)}{dx^2} - (2 - \frac{x}{2}) - \int_0^1 xy(t)y^1(t)dt \qquad (37)$$

Thus, collocating (37) at the points $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = {}^3/_4$ and utilizing the given conditions, we obtained system of nonlinear algebraic equations which were also solved by Newton's method. The following values were obtained:

$a_0 = 2$, $a_1 = 0$, $a_2 = 0$, $c_1 = 0$ and $c_2 = 0$

Substituting these values into (36), the result will be as y (x) = x^2, that is the exact solution.

Example 5: Consider the nonlinear FIDE

$$y^{//}(x) + x\,y^1(x) - x\,y(x) = e^x - sin(x) + \int_0^1 sin(x)e^{-2t}y^2(t)dt, \quad 0 \le x,t \le 1$$

with conditions: $y(0) = 1, \quad y^/(0) = 1$

and the exact solution is $y(x) = \exp(x)$

Using the proposed method the first order approximate series solution for N=4 can be expressed as

$y(x)=1+x+0.5001766022x^2+0.1685402034x^3+0.04367390939x^4+0.005029212484x^5+0.003521747354x^6+0.000450084323x^7+0.0001201260022x^8-0.00003366241046x^9-4.0362278 \times 10^{-10} x^{11} + 2.58732546 \times 10^{-12} x^{13}$

The approximate series solution of Example 5 is compared with the exact solution, and the solution of Ordokhani and Dehestani (2013) in Table 2 which shows that the method is quite efficient.

Table 2: Numerical results of Example 5

X	Exact Solution	IC-HAM Solution	Error of IC-HAM N=4	Error of Ordokhani and Dehestani (2013).

				N=4
0	1.00000000	1.000000000	0.00	0.00
0.1	1.105170918	1.105174727	3.81E-06	-
0.2	1.221402758	1.221427092	2.43E-05	2.68E-04
0.3	1.349858808	1.349934935	7.61E-05	-
0.4	1.491824698	1.491998138	1.73E-04	2.15E -03
0.5	1.648721271	1.649050372	3.29E-04	-
0.6	1.822118800	1.822672863	5.54E-04	7.15E-03
0.7	2.013752707	2.014610022	8.57E-04	-
0.8	2.225540928	2.226786865	1.25E-03	1.61E -02
0.9	2.459603111	2.461328003	1.72E-03	-
1.0	2..7182818258	2.720578053	2.30E-03	2.90E-02

Example 6: Consider the Nonlinear Volterra IDE

$$y^{//}(x)+\int_0^x (y(t))^2 dt+\left(\frac{x}{2}-S\,\text{inh}\,x-\frac{1}{4}\text{Sinh}\,2x\right)=0,\quad 0\le s,x\le 1$$

subject to the boundary conditions
y(0) = 0, y(1) = Sinh(1)
The exact solution is $y(x)=\sinh(x)$
Similarly, using the proposed method for case N=5, we obtain the following approximate solution:
$y(x)$=1.000000335514422x-0.000003751688x^2+0.1666876846x^3-0.00006543902x^4+ 0.0008450551855x^5-0.0001162240987x^6+0.000250889928x^7-1.966201000X$10^{-7}x^8$ +0.000002516215704x^9–1.765716171X$10^{-7}x^{10}$+4.776504870X$10^{-8}x^{11}$-1.508840627X$10^{-8}x^{12}$+ 2.290840760X$10^{-9}x^{13}$-4.531962600X$10^{-10}x^{14}$+1.382069651X$10^{-9}x^{15}$–7.766972500X$10^{-12}x^{16}$+ 1.236412037X $10^{-11}x^{17}$

In Table 3, the numerical results of Example 6 are presented and the results are also compared with the exact solutions. It is observed that the method is quite efficient and accurate.

Table 3: Numerical results of Example 6

X	Exact Solution	IC-HAM Solution	Error
0	0.000000000	0.000000000	0.00
0.1	0.1001667500	0.1001667617	1.17E-08
0.2	0.2013360025	0.2013360139	1.14E-08
0.3	0.3045202934	0.3045203055	1.21E-08
0.4	0.4107523258	0.4107523386	1.28E-08
0.5	0.5210953055	0.5210953187	1.32E-08
0.6	0.6366535821	0.6366535960	1.39E-08
0.7	0.7585837018	0.7585837163	1.45E-08
0.8	0.8881059822	0.8881059966	1.44E-08
0.9	1.0265167257	1.026516741	1.50E-08
1.0	1.1752011936	1.175201194	4.00E-10

Conclusion

The main objective of this paper is to introduce a new method for solving nonlinear integro-differential equations. We have achieved this objective by using the coupling technique of integral collocation method and homotopy analysis method. One major advantage of this method is that the problem of finding the value of convergence-control parameter does not arise. In order to illustrate the accuracy of the method, we compared our approximate solutions of the examples with their exact solutions and approximate solutions presented by other methods. The numerical results showed that the proposed method can solve nonlinear integro-differential equations effectively and the comparison shows that the results of the new method are in good agreement with the existing results in the literature.

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